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Generalization of Bochner's theorem for functions of the positive type

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Abstract

We generalize Bochner's theorem for functions of the positive type—theorem 1—to more general integral transforms using the Jost solution of the radial Schrödinger equation. The generalized theorem is theorem 2. We then use Bochner's theorem to obtain an integral representation for the phase shift, shown in theorem 4. In a forthcoming paper, this theorem will be used in inverse scattering theory. The proofs are simple, and make use of well-known theorems of real analysis and Fourier transforms of L^1, $L^1 \cap L^2, \ldots$ functions.

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1. Introduction

In a recent paper [1][2], we generalized a simple theorem of Titchmarsh on the positivity of Fourier sine transforms (cf [2][3]) to more general integral transforms where the sine function is replaced by the appropriate regular solution of the radial Schrödinger equation (cf [3–6]). In the present paper, using similar techniques and restricting ourselves to the case of functions with support on the positive half-axis, we generalize Bochner's theorem on the Fourier integral transforms of functions of the positive type (cf [7–9]) to more general integral transforms in which the exponential function is replaced by the Jost solution of the radial Schrödinger equation (cf [3–6]).

Bochner's theorem reads as follows.

Theorem 1. *If $\alpha(t)$ is a non-decreasing bounded function on $(-\infty, \infty)$, and if $F(x)$ is defined by the Stieltjes integral*

$$F(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt}\, \mathrm{d}\alpha(t), \qquad -\infty < x < \infty, \tag{1}$$

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[2] See specially appendix B.
[3] For a very large number of explicit examples, see [12].

then $F(x)$ is a continuous function of the positive type. Conversely, if $F(x)$ is measurable on $(-\infty, \infty)$, and F is of the positive type, then there exists a non-decreasing bounded function $\alpha(t)$ such that $F(x)$ is given by (1) for almost all x, $-\infty < x < \infty$.

We should remark that, in the converse part of the theorem, Bochner assumed $F(x)$ to be continuous, and showed that $\alpha(t)$ is such that (1) is true for all x. Riesz showed that measurability of $F(x)$ was sufficient in the converse theorem.

We recall the reader that a (not necessarily measurable) function $F(x)$ defined on $(-\infty, \infty)$ is said to be of the positive type if

$$\sum_{m=1}^{s}\sum_{n=1}^{s} a_m \overline{a}_n F(x_m - x_n) > 0 \tag{2}$$

for any finite number of arbitrary real $x_1, \ldots, x_s$ and a like number of complex $a_1, \ldots, a_s$.

Remark. As we said before, we consider the case where the support of $\alpha(t)$ is restricted to the half-axis $t \geqslant 0$ in (2). It is then obvious that $F(x)$ can be extended analytically in the upper half-plane $\mathrm{Im}\, x > 0$, and it is holomorphic and bounded there.

The theorem which generalizes the theorem of Bochner in the present case is now as follows.

Consider the Stieltjes integral

$$\widetilde{f}(k) = \int_0^\infty f(k,r)\,\mathrm{d}\alpha(r), \tag{3}$$

where $f(k,r)$ is the (Jost) solution defined by (cf [3-6])

$$\begin{cases} f''(k,r) + k^2 f(k,r) = V(r) f(k,r), \\ r \in [0,\infty), \quad V(r) > 0, \quad V \in L^1(0,1), \quad rV \in L^1(1,\infty), \\ \lim\limits_{r\to\infty} \mathrm{e}^{-\mathrm{i}kr} f(k,r) = 1, \\ \lim\limits_{r\to\infty} \mathrm{e}^{-\mathrm{i}kr} f'(k,r) = \mathrm{i}k. \end{cases} \tag{4}$$

For each fixed value of r ($\geqslant 0$), $f(k,r)$ is holomorphic and bounded for k in $\mathrm{Im}\, k > 0$. It vanishes exponentially there as $|k| \to \infty$ in all directions not parallel to the real axis. For k real, $f(k,r)$ is simply bounded.

The purpose of the present paper is to extend, as much as possible, the results of theorem 1 to the more general integral transform (3). When $V(r) \equiv 0$, $f(k,r) \equiv \mathrm{e}^{\mathrm{i}kr}$, and we should recover theorem 1. The general form of the theorem we are looking for is given in theorem 2 below. It needs first the introduction of the Marchenko kernel (cf [5][4]).

2. Generalization of theorem 1 to (3)

The Jost solution defined by (4) has the integral representation [5, chapter 5] and [6, chapter 4]:

$$f(k,r) = \mathrm{e}^{\mathrm{i}kr} + \int_r^\infty A(r,t)\,\mathrm{e}^{\mathrm{i}kt}\,\mathrm{d}t, \tag{5}$$

[4] This book contains full references to original works.

where, for each fixed $r \geqslant 0$ and $t \geqslant r$, the kernel $A(r,t) \in L^1(r,\infty) \cap L^2(r,\infty)$ in t is the solution of the integral equation

$$A(r,t) = \frac{1}{2}\int_{\frac{r+t}{2}}^{\infty} V(s)\,\mathrm{d}s + \int_{\frac{r+t}{2}}^{\infty} \mathrm{d}s \int_0^{\frac{t-r}{2}} V(s-u)A(s-u,s+u)\,\mathrm{d}u. \tag{6}$$

It can be shown that this integral equation has a unique positive solution obtained by iteration (absolutely convergent series !) and satisfies the bound ($V(r) > 0$!):

$$0 < A(r,t) \leqslant \frac{1}{2}\int_{\frac{r+t}{2}}^{\infty} V(s)\,\mathrm{d}s \left[\exp\int_r^{\infty} uV(u)\,\mathrm{d}u\right] \leqslant C\int_{\frac{r+t}{2}}^{\infty} V(s)\,\mathrm{d}s, \tag{7}$$

where C is an appropriate constant.

Note here that, according to (4), the integrals are absolutely convergent. Also, it is obvious on (7) that $A(r,t)$ is a bounded continuous function, and goes to zero at infinity when $r\ (\leqslant t) \to \infty$ or $t \to \infty$. We can then replace in (3) $f(k,r)$ by its integral representation (5), and exchange the order of integrations (cf [2, 7, 8]), to find

$$\begin{cases} \tilde{f}(k) = \displaystyle\int_0^{\infty} \mathrm{e}^{\mathrm{i}kr}\,\mathrm{d}\beta(r), & (8a) \\ \mathrm{d}\beta(r) = \mathrm{d}\alpha(r) + \left(\displaystyle\int_0^r A(t,r)\,\mathrm{d}\alpha(t)\right)\mathrm{d}r. & (8b) \end{cases}$$

$A(t,r)$ being positive, it follows that $\beta(r)$ is bounded and increasing, if $\alpha(r)$ is so. Consider now a function $\tilde{f}(k)$ of positive type. From the second part of Bochner's theorem, we have representation (8a), where $\beta(r)$ is positive, bounded and non-decreasing. Once the existence of $\beta(r)$ is shown, one has to solve the Volterra integral equation (cf [11])

$$\alpha'(r) = \beta'(r) - \int_0^r A(t,r)\alpha'(t)\,\mathrm{d}t. \tag{9}$$

The kernel $A(t,r)$ being a bounded continuous function, it is known, as usual with Volterra integral equations, that (9) has a unique solution obtained by iteration, i.e. iterating (9), by starting from $\beta'(r)$, we obtain an absolutely and uniformly convergent series defining the solution (cf [11]). Moreover, since $A(t,r) \to 0$ as $r \to \infty$, all the higher terms of the series go to zero. In fact, they are all $L^1(\infty)$ in r since

$$\int_r^{\infty} V(s)\,\mathrm{d}s \in L^1(1,\infty). \tag{10}$$

The first term of the series, namely $\beta'(r)$ being itself L^1, it is obvious that the solution $\alpha'(r)$, given by an absolutely and uniformly convergent series, is also L^1. Since both $\beta(r)$ and $A(t,r)$ are positive, and just looking at the sign of each term in (8b) or (9), we see immediately that the solution $\alpha(r)$ is such that

$$\begin{cases} \text{i) if } \alpha'(r) > 0, \text{ then } \alpha'(r) < \beta'(r), \\ \text{ii) } \alpha'(r) < 0 \text{ is impossible}, \\ \text{ii) } \alpha'(r) \text{ can be oscillating.} \end{cases} \tag{11}$$

Putting together everything, we have the generalized Bochner's theorem.

Theorem 2. *Consider the Stieltjes integral (3), with $\alpha(r)$ positive, bounded and non-decreasing. Then $\tilde{f}(k)$ is a function of positive type having the usual representations (8a), (8b). Conversely, if we consider a function $\tilde{f}(k)$, holomorphic and bounded in* $\mathrm{Im}\,k > 0$, *and of positive type, then it can be represented in the form (3), where $\alpha(r)$, not necessarily positive or non-decreasing, is given by the unique solution of the Volterra integral equation (9). The* *kernel $A(t, r)$ itself is defined by the unique solution of the integral equation (6), $V(r)$ being the potential defining the integral representation (3) via (4).*

Remark. Note the unsymmetry between $\alpha(r)$ and $\beta(r)$. If $\alpha(r)$ is non-decreasing, so is $\beta(r)$. However, the converse is not true, as seen in (9), unless $A(t, r)$ is small, so that, in the iteration of (9), the dominant term is $\beta'(r)$. And for $A(t, r)$ to be small enough, one sees in (7) that $V(r)$, positive, must be small enough.

3. Application of theorem 1 to the phase shift

We consider again the radial Schrödinger equation for the S-wave with a positive potential satisfying

$$\int_0^\infty rV(r)\,\mathrm{d}r < \infty. \tag{12}$$

The regular wavefunction $\varphi(k, r)$ satisfies the same differential equation shown in (4), but now with the conditions

$$\varphi(k, 0) = 0, \qquad \varphi'(k, 0) = 1. \tag{13}$$

Obviously, $\varphi(k, r)$ is an even function of k, and is, in fact, an entire function of k (for each fixed r) of exponential type: order 1 and type r [3-6]. For $|k| \to \infty$, it has the asymptotic form $\varphi \sim \frac{\sin kr}{k} + \cdots$.

In general, it can be shown that the phase shift $\delta(k)$ (cf [3, 4]) can be written as (cf [5, 6])

$$\begin{cases} \delta(k) = -\displaystyle\int_0^\infty \gamma(t)\sin kt\,\mathrm{d}t, \\ \gamma(t) \text{ real and } \in L^1(0, \infty). \end{cases} \tag{14}$$

The minus sign in front of the integral is just for convenience. Clearly, $\delta(k)$ is a continuous and bounded function of k for all $k \geqslant 0$, and vanishes at $k = 0$ and $k = \infty$. Moreover, one can show that (cf [5, 6])

$$\frac{\delta(k)}{k} \in L^1(0, \infty). \tag{15}$$

Another representation of the phase shift is the following (cf [13]):

$$\delta(k) = -k\int_0^\infty V(r)\frac{\varphi^2(k, r)}{\varphi'^2(k, r) + k^2\varphi^2(k, r)}\,\mathrm{d}r. \tag{16}$$

The integral is shown to be absolutely convergent. This representation is found to be very useful for proving many properties of the phase shift from a single formula. Its general form for all angular momenta is also given in (cf [13]).

Remark. For k real and $\neq 0$, the fraction under the integral

$$\phi(k, r) = \frac{\varphi^2(k, r)}{\varphi'^2(k, r) + k^2\varphi^2(k, r)} \tag{17}$$

is always a bounded function for all $k > 0$, and all $r \geqslant 0$. Indeed, for $k > 0$, φ and φ' cannot both vanish simultaneously for some $r = r_0$ without having $\varphi \equiv 0$ (cf [10]). For $r = 0$, the denominator is just 1, by the definition. For $k = 0$, the denominator reduces to $\varphi'^2(0, r)$, and because of $V(r) \geqslant 0$, $\varphi'(0, r)$ is an increasing function of r, starting from $\varphi'(0, r = 0) = 1$

(cf [1, 10]). Moreover, because of (5), we have, for each r ($\geqslant 0$) fixed,

$$\phi(k,r) = \frac{\sin^2 kr}{k^2} + \cdots, \qquad k \to \pm\infty \tag{18}$$

so that $\phi(k,r) \in L^1(0,\infty)$ in the variable k.

From formula (16), one can show that $\delta(k)$ is a differentiable function of k for all $k > 0$ (cf [13]). In order to secure also the differentiability at $k = 0$, one needs to impose the extra condition at infinity:

$$r^2 V(r) \in L^1(1,\infty). \tag{19}$$

We can summarize the above results in the following.

Theorem 3. *Under condition (12) on the potential, the phase shift $\delta(k)$ is a continuous and bounded function of k for all $k \geqslant 0$ and satisfies (15). It is also continuously differentiable for all $k > 0$. If (19) is also satisfied, the derivative exists also for $k = 0$, and is finite. Obviously, $\delta(0) = \delta(\infty) = 0$.*

We introduce now

$$\Gamma(t) = \int_t^\infty \gamma(u)\,\mathrm{d}u. \tag{20}$$

By the definition, $\Gamma(t)$ is a bound and continuous function of t for all $t \geqslant 0$ and $\Gamma(\infty) = 0$. Using now $\gamma(t) = -\Gamma'(t)$ in (14) and integrating by parts, we find

$$\delta(k) = -k \int_0^\infty \Gamma(t)\cos kt\,\mathrm{d}t, \tag{21}$$

the integral being convergent at infinity by the Abel lemma (cf [2]). Comparing (21) with (15) and (16), and inverting the Fourier cosine transform, we get

$$\begin{aligned}\Gamma(t) &= \frac{2}{\pi}\int_0^\infty \frac{-\delta(k)}{k}\cos kt\,\mathrm{d}k = \frac{2}{\pi}\int_0^\infty\left[\int_0^\infty V(r)\frac{\varphi^2}{\varphi'^2 + k^2\varphi^2}\,\mathrm{d}r\right]\cos kt\,\mathrm{d}k \\ &= \frac{2}{\pi}\int_0^\infty V(r)\,\mathrm{d}r\int_0^\infty \frac{\varphi^2(k,r)}{\varphi'^2 + k^2\varphi^2}\cos kt\,\mathrm{d}k,\end{aligned} \tag{22}$$

the exchange of the two integrations being allowed by virtue of the remark after (16), i.e. (18).

For each fixed $r \geqslant 0$ $\phi(k,r)$, defined by (17), is a real, bounded and continuous function of k for all $k \geqslant 0$ and vanishes at $k = \pm\infty$. Obviously, it is also positive and even in k. Therefore, it is straightforward to show that ϕ is a function of the positive type as was defined in (2). It follows that, according to theorem 1, for each r fixed ($\geqslant 0$), we have

$$\phi(k,r) = \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}kt}\,\mathrm{d}\alpha(r,t), \tag{23}$$

with α being a bounded non-decreasing function of t. However, $\phi(k,r)$, satisfying (18), is $L^1(0,\infty)$ in k. The inversion of (23) then shows that, in fact $\dot\alpha(r,t) \equiv \mathrm{d}\alpha(r,t)/\mathrm{d}t$ is continuous and bounded and vanishes at $t = \pm\infty$. Using now the fact that $\phi(k,r)$ is a real even function of k, we can write (23) as

$$\begin{cases}\phi(k,r) = \displaystyle\int_0^\infty \omega(r,t)\cos kt\,\mathrm{d}t, \\ \omega(r,t) = \displaystyle\frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}t}\alpha(r,t) + \frac{\mathrm{d}}{\mathrm{d}t}\alpha(r,-t)\right] > 0,\end{cases} \tag{24}$$

with $\omega(r,t)$ being a bounded and continuous function of t, and

$$\omega(r,\infty) = 0. \tag{25}$$

Since, according to (18), $\phi(k,r)$ is L^1 in k, we can invert (24) and write

$$\omega(r,t) = \frac{2}{\pi}\int_0^\infty \phi(k,r)\cos kt\,\mathrm{d}k, \tag{26}$$

the integral being absolutely convergent. Therefore, $\omega(r,t)$, for each $r \geqslant 0$, is a continuous and bounded function of t for all $t \geqslant 0$. This, used now in (22), leads to

$$\Gamma(t) = \frac{2}{\pi}\int_0^\infty \omega(r,t)V(r)\,\mathrm{d}r > 0, \tag{27}$$

since both ω and V are positive. We should remark here that, from the definitions (17) and (26), it follows from (13) that $\omega(r,t)$ is $0(r^2)$ as $r \to 0$. Also, we have (25). Therefore, the integral in (27) is absolutely convergent and defines a bounded function of t for all $t \geqslant 0$. We can therefore summarize our result in the following.

Theorem 4. *Under assumption (12) on the positive potential $V(r)$, the phase shift has the integral representation (21), where $\Gamma(t)$—a continuous and bounded function according to its definition (20), and vanishing at $t = \infty$—is positive.*

We shall give an application of this theorem to inverse scattering problem in a forthcoming paper.

Remark. Formula (16) is quite general, and is valid for all V satisfying $rV(r) \in L^1(0,\infty)$, whether positive or not. It shows the well-known fact that the phase shift has the opposite sign of V in cases where V has a definite sign. When $V(r)$ is negative, and admits some bound states of energies $-\gamma_j^2$, $j = 1,\ldots,n$, one can show that (cf [3, 5])

$$\widetilde{\delta}(k) = \delta(k) - 2\sum_j \mathrm{Arctg}\frac{\gamma_j}{k} \tag{28}$$

has a similar representation as (14), and one has, of course, $\widetilde{\delta}(0) = \widetilde{\delta}(\infty) = 0$. One may then be tempted to apply our theorem 4 to $\widetilde{\delta}(k)$. However, it is not obvious that $\widetilde{\delta}(k)$ corresponds to a positive potential $\widetilde{V}(r)$. The corresponding potential may be oscillating, while being weak enough not to admit bound states.

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